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Analysis about Optimal Portfolio under G-Expectation

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Abstract

In this paper, an optimal portfolio selection rule under G-expectation is established and explicit optimal portfolio for a particular class of utility functions is investigated.

Keyword: G-Brownian motion; G-expectation; optimal portfolio; HJB equation

1. Introduction

The theory about Ito calculus plays an important role in the development of Financial Economics, for example, Black and Scholes (1973) [1]. Starting with the seminal work by Merton (see[3] and [4]), there has been a lot of research on continuous-time portfolio optimization. Motivated by uncertainty problems, risk measures and the superhedging in finance, Peng (2006) [5] has introduced the notion of sublinear expectation space, which is a generalization of probability space.

Together with the notion of sublinear expectation, Peng (2010) [6] also introduced the related G-normal distribution and G-Brownian motion. The expectation associated with G-Brownian motion is a sublinear expectation which is called G-expectation. G-Brownian motion has a very rich and interesting new structure which nontrivially generalizes the classical one. Since these notions were introduced, many properties of G-expectation have been studied in Denis, Hu and Peng (2010) [2], et al.

In this paper, we investigate the two assets and one-period optimal portfolio selection problem in the framework of sublinear expectation space. Following Black and Scholes (1973) [1], we assume that the stock price follows the stochastic differential equation driven by G-Brownian motion, specifically, so-

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called geometric G-Brownian motion. It can be used to describe the volatility uncertainty of stock price. By using the dynamic programming technique, we established the optimal portfolio selection model under volatility uncertainty, explicit optimal portfolio for a particular class of utility functions is presented. we find that the optimal portfolio depends on the maximal and minimal volatility of stock price.

2. Preliminaries

In this section, we introduce some notations and preliminaries about G-Brownian motion and G-stochastic integrals, which will be needed in what follows. More details of this section can be founded in Peng (2006) [5] and Peng (2010) [6].

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued bounded functions defined on Ω. We suppose that $\mathcal{H}$ satisfies $C \in \mathcal{H}$ for each constant C and $|X| \in \mathcal{H}$, if $X \in \mathcal{H}$.

Definition 2.1. A sublinear expectation $\mathbb{E}$ is a functional $\mathbb{E}: \mathcal{H} \to R$ satisfying

(i) Monotonicity: $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \geq Y$. (ii) Constant preserving: $\mathbb{E}[C] = C$ for $C \in R$.

(iii) Sub-additivity: For each $X, Y \in \mathcal{H}, \mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.

(iv) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. If (i) and (ii) are satisfied, $\mathbb{E}$ is called a nonlinear expectation and the triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a nonlinear expectation space. The notions and properties about identical distribution, independence, G-normal distribution ($N(0;[\underline{\sigma}^2, \bar{\sigma}^2])$), G-expectations and G-Brownian motion as well as the definition of spaces $L_G^p(\Omega_T)$, $M_G^{p,0}(0,T)$ and $M_G^p(0,T)$ can be found in Peng (2010) [6].

Let $(B_t)_{t\geq 0}$ be a 1-dimensional G-Brownian motion with $G(a) := \frac{1}{2}\mathbb{E}[aB_1^2] = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$, where $\bar{\sigma}^2 = \mathbb{E}[B_1^2], \underline{\sigma}^2 = -\mathbb{E}[-B_1^2]$, $0 \leq \underline{\sigma} \leq \bar{\sigma} < \infty$. The notions and properties about $\int_0^t \eta_s ds$, $\int_0^t \eta_s dB_s$, $\int_0^t \eta_s d\langle B\rangle_s$ can be found in Peng [5] and [6].

The fundamental tool for formal manipulation and solution of stochastic processes of the G-Ito type is G-Ito's Lemma stated as follows.

Lemma 2.1 (See Peng (2010) [6]). Let $\Phi \in C^2(R)$ with $\Phi_x, \Phi_{xx} \in C_{b,Lip}(R)$. Let $t \in [0,T]$ be fixed and let X_t be a 1-dimensional process on [0,T] of the form

$$X_t = x + \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B\rangle_s + \int_0^t \beta_s dB_s,$$

where α, η, β are bounded processes in $M_G^2(\Omega_t)$. Then for each $t \geq 0$ we have, in $L_G^2(\Omega_t)$

$$\Phi(X_t) - \Phi(x) = \int_0^t \Phi_x(X_u)\alpha_u du + \int_0^t \Phi_x(X_u)\beta_u dB_u + \int_0^t [\Phi_x(X_u)\eta_u + \frac{1}{2}\Phi_{xx}\beta_u^2] d\langle B\rangle_u.$$

Before proceeding to the discussion of optimal portfolio selection problem, another concept useful for working with Ito Processes is the nonlinear differential generator of the stochastic process X_t.

Lemma 2.2. Let $f \in C_b^2(R)$, in $L_G^1(\Omega_t)$, the nonlinear differential generator of the stochastic process X_t is defined by $A_G f(X_t) = \lim_{h \downarrow 0} \frac{\mathbb{E}[f(X_{t+h}) - f(X_t) | \Omega_t]}{h}$, then

$$A_G f(X_t) = \sup_{\underline{\sigma} \le r \le \overline{\sigma}} \{ f_x(X_t)[\alpha_t + \eta_t r^2] + \frac{1}{2} f_{xx}(X_t)\beta^2 r^2 \}.$$

3. Optimal portfolio selection model under volatility uncertainty

In order to express the essence of our model, we only consider two-asset and one-period model of portfolio selection, but our model can be extended to the multi-asset and multi-period model. The multi-asset and multi-period cases are much more complicated and we hope to study these cases in a forthcoming publication. Let X_t denote the wealth of a person at time t. Suppose that the person has the choice of two different investments, for example, stock and bond. The stock price $S(t)$ at time t is assumed to satisfy the equation $dS(t) = S(t)(\alpha dt + \beta d\langle W \rangle_t + \gamma dW_t)$, where W_t denotes G-Brownian motion (see Peng (2006) or (2010)), and α, β, γ are constants, we call it geometric G-Brownian motion.

The investment is called risky, since $\gamma > 0$. We assume that the bond price $B(t)$ satisfies a similar equation, but with no random term: $dB(t) = B(t)rdt.$ This investment is called safe. So it is natural to assume $r < \alpha$. At each instant the investor can choose how big fraction u of his wealth he will invest in the risky asset, thereby investing the fraction $1-u$ in the safe one. This gives the following stochastic differential equation for the wealth $X_t = X_t^u$:

$$\begin{aligned} dX(t) &= (1-u)X_t r dt + uX_t(\alpha dt + \beta d\langle W \rangle_t + \gamma dW_t) \\ &= X_t(\alpha u + r(1-u)dt + u\beta d\langle W \rangle_t) + \gamma u X_t dW_t. \end{aligned}$$

Suppose that, starting with the wealth $X_t = x > 0$ at time t, the investor wants to maximize the expected utility of the wealth at some future time T>t. If we allow no borrowing (i.e. $X \ge 0$) and are given a utility function $U:[0,\infty) \to [0,\infty), U(0) = 0$ (usually assume to be increasing and concave), the problem is to find $V(t,x)$ and a control $u^* = u^*(t, X_t), 0 \le u^* \le 1$, $V(t,x) = \sup J^u(t,x) = J^{u^*}(t,x)$, where $J^u(s,x) = \mathbb{E}^x[U(X_{t_1}^u)]$, and t_1 is the first exit time from the region $D = \{(r,z): r < T, z > 0\}$. The differential operator L^v has the form

$$(L^v f)(t,x) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} x(\alpha v + r(1-v)) + 2G(\frac{\partial f}{\partial x} xv\beta + \frac{1}{2}\gamma^2 v^2 x^2 \frac{\partial^2 f}{\partial x^2}), \qquad (1)$$

where $G(a) := \frac{1}{2} \sup_{\underline{\sigma} \le \sigma \le \overline{\sigma}} (a\sigma^2).$

The HJB equation become $\sup\{L^v V(t,x)\} = 0, (t,x) \in D,$ and $V(T,x) = U(x)$, $V(t,0) = U(0)$ for t<T. Therefore, for each (t,x) we try to find the value $v = u(t,x)$ which maximizes the function

$$(L^v V)(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} x(r + (\alpha - r)v) + 2G(\frac{\partial V}{\partial x} xv\beta + \frac{1}{2}\gamma^2 v^2 x^2 \frac{\partial^2 V}{\partial x^2}). \qquad (2)$$

If $\frac{\partial V}{\partial x} > 0$ and $\frac{\partial^2 V}{\partial x^2} < 0$, as well as $v \ge -\frac{2\beta \frac{\partial V}{\partial x}}{\gamma^2 x \frac{\partial^2 V}{\partial x^2}}$, then the solution is

$$\underline{v} = \underline{u}(t,x) = -\frac{(\alpha - r + \beta\underline{\sigma}^2)\frac{\partial V}{\partial x}}{\gamma^2 x \underline{\sigma}^2 \frac{\partial^2 V}{\partial x^2}}. \tag{3}$$

If we substitute this into the HJB (2) we get the following nonlinear boundary value problem for $V(t,x)$:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} x\Big(r - \frac{(\alpha - r)(\alpha - r + \beta\underline{\sigma}^2)\frac{\partial V}{\partial x}}{\gamma^2 x \underline{\sigma}^2 \frac{\partial^2 V}{\partial x^2}}\Big) + \frac{(\alpha - r + \beta\underline{\sigma}^2)(\alpha - r - \beta\underline{\sigma}^2)(\frac{\partial V}{\partial x})^2}{2\underline{\sigma}^2\gamma^2 \frac{\partial^2 V}{\partial x^2}} = 0,$$

namely

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x} xr - \dfrac{(\alpha - r + \beta\underline{\sigma}^2)^2(\frac{\partial V}{\partial x})^2}{2\underline{\sigma}^2\gamma^2 \frac{\partial^2 V}{\partial x^2}} = 0 \quad (t,x) \in (0,T) \times (0,\infty), \\ V(T,x) = U(x), V(t,0) = U(0). \end{cases} \tag{4}$$

On the other hand, if $\frac{\partial V}{\partial x} > 0$ and $\frac{\partial^2 V}{\partial x^2} < 0$, as well as $v < -\frac{2\beta \frac{\partial V}{\partial x}}{\gamma^2 x \frac{\partial^2 V}{\partial x^2}}$, then the solution is

$$\overline{v} = \overline{u}(t,x) = -\frac{(\alpha - r + \beta\overline{\sigma}^2)\frac{\partial V}{\partial x}}{\gamma^2 x \overline{\sigma}^2 \frac{\partial^2 V}{\partial x^2}}. \tag{5}$$

If we substitute this into the HJB (1.2), we get the following nonlinear boundary value problem for V:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x} xr - \dfrac{(\alpha - r + \beta\overline{\sigma}^2)^2(\frac{\partial V}{\partial x})^2}{2\overline{\sigma}^2\gamma^2 \frac{\partial^2 V}{\partial x^2}} = 0 \quad (t,x) \in (0,T) \times (0,\infty), \\ V(T,x) = U(x), V(t,0) = U(0). \end{cases} \tag{6}$$

The problems (4) and (6) are hard to solve for general U, but we can solve them for particular utility function, for example, the power functions and logarithmic functions.

4. Explicit solutions for a particular class of utility functions

We suppose that the stock price follows a geometric G-Brownian motion. If a further assumption about the utility function of the individual is made, then the systems (4) and (6) can be solved in closed form, and the optimal portfolio rules derived explicitly. Assume that the utility function for the individual, $U(x)$ can be written as $U(x) = x^{\lambda}$, where $0 < \lambda < 1$. If we choose such a utility function U, we try to find solutions of (4) and (6) of the form $\phi(t,x) = f(t)x^{\lambda}$. Substituting we obtain

$$\underline{\phi}(t,x)=\exp\{(r\lambda+\frac{(\alpha-r+\beta\underline{\sigma}^2)^2\lambda}{2\underline{\sigma}^2\gamma^2(1-\lambda)})(T-t)\}x^\lambda,$$

and $\overline{\phi}(t,x)=\exp\{(r\lambda+\frac{(\alpha-r+\beta\overline{\sigma}^2)^2\lambda}{2\overline{\sigma}^2\gamma^2(1-\lambda)})(T-t)\}x^\lambda$, are the solutions of (4) and (6), respectively.

Using (3) and (5) we obtain the optimal controls

$\underline{u}^*(t,x)=\max\{\frac{\alpha-r+\beta\underline{\sigma}^2}{\underline{\sigma}^2\gamma^2(1-\lambda)},\frac{2\beta}{\gamma^2(1-\lambda)}\}$ and $\overline{u}^*(t,x)=\min\{\frac{\alpha-r+\beta\overline{\sigma}^2}{\overline{\sigma}^2\gamma^2(1-\lambda)},\frac{2\beta}{\gamma^2(1-\lambda)}\}$. If $\max\{\frac{\alpha-r+\beta\underline{\sigma}^2}{\underline{\sigma}^2\gamma^2(1-\lambda)},\frac{2\beta}{\gamma^2(1-\lambda)}\}\in(0,1)$, and $\min\{\frac{\alpha-r+\beta\overline{\sigma}^2}{\overline{\sigma}^2\gamma^2(1-\lambda)},\frac{2\beta}{\gamma^2(1-\lambda)}\}\in(0,1)$, then they are the optimal solutions of the problems (4) and (6). Note that $\underline{u}^*$ and $\overline{u}^*$ are in fact constants.

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